

On the mod-Gaussian convergence of a sum over primes

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Abstract We prove mod-Gaussian convergence for a Dirichlet polynomial which approximates $\operatorname{Im} \log \zeta(1/2 + it)$. This Dirichlet polynomial is sufficiently long to deduce Selberg's central limit theorem with an explicit error term. Moreover, assuming the Riemann hypothesis, we apply the theory of the Riemann zeta-function to extend this mod-Gaussian convergence to the complex plane. From this we obtain that $\operatorname{Im} \log \zeta(1/2 + it)$ satisfies a large deviation principle on the critical line. Results about the moments of the Riemann zeta-function follow.

Keywords Distribution of primes · Mod-Gaussian convergence · Riemann zeta-function · Selberg's central limit theorem · Large deviations

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1 Introduction

In this paper we study the distribution of values taken by $\log \zeta(1/2 + it)$. A breakthrough was achieved by Selberg who showed that as t varies in $[T, 2T]$, the distribution of $(\operatorname{Re} \log \zeta(1/2 + it), \operatorname{Im} \log \zeta(1/2 + it))$ is approximately Gaussian, with independent components each having expectation 0 and variance $(\log \log T)/2$. More precisely, he proved a central limit theorem which, by the Lévy continuity theorem, is equivalent to the statement that

$$\frac{1}{T} \int_T^{2T} e^{iu \frac{\operatorname{Re} \log \zeta(1/2 + it)}{\sqrt{(\log \log T)/2}} + iv \frac{\operatorname{Im} \log \zeta(1/2 + it)}{\sqrt{(\log \log T)/2}}} dt \rightarrow e^{-u^2/2 - v^2/2}, \quad (1.1)$$

as $T \rightarrow \infty$, for all real numbers u and v . For the case of $\operatorname{Im} \log \zeta(1/2 + it)$ see [17, 18], and also the work of Ghosh [7]. The general case is investigated for instance in the book of Joyner

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[11]. Some of Selberg's more recent results, for example about the rate of convergence, can be found in [19] and the thesis of Tsang [21]. Initially, Selberg obtained the asymptotics of the joint moments which lead to (1.1) by the method of moments. A more effective approach, applied in our analysis, too, is treated in the work of Bombieri and Hejhal [2]. A central limit theorem for the sum over primes $(1/\sqrt{(\log \log x)/2}) \sum_{p \leq x} p^{-1/2-iU_T}$, U_T being random variables uniformly distributed on $[T, 2T]$, $\log x = \log T/(\log \log T)^{1/4}$, follows from the mean value theorem of Montgomery and Vaughan and the method of moments. To complete the proof (see [2, Lemma 3 and Corollary]), they showed that the L^1 -norm of $\log \zeta(1/2 + iU_T) - \sum_{p \leq x} p^{-1/2-iU_T}$ is sufficiently small.

The convergence in (1.1) is also a consequence of a conjecture on the behaviour of the moments of the Riemann zeta-function on the critical line (see, e.g., the work of Keating and Snaith [12] and the references therein). It asserts that

$$\begin{aligned} & e^{(z_1^2+z_2^2)(\log \log T)/4} \frac{1}{T} \int_T^{2T} e^{iz_1 \operatorname{Re} \log \zeta(1/2+it)+iz_2 \operatorname{Im} \log \zeta(1/2+it)} dt \\ & \rightarrow \Phi_g(z_1, z_2) \Phi_a(z_1, z_2) \quad \text{as } T \rightarrow \infty \end{aligned} \quad (1.2)$$

locally uniformly for $z_1, z_2 \in \mathbb{C}$ with $\operatorname{Re}(iz_1) > -1$ and analytic functions Φ_g, Φ_a (see [13, Conjecture 9] and also [9, Conjecture 1]). This type of convergence was introduced in [10] where it is called mod-Gaussian convergence.

A precise form of the function Φ_g was conjectured by Keating and Snaith and is based on calculations in the theory of random matrices (see [12], [13, formula (18)]). The arithmetic factor Φ_a can be explained, e.g., by computing the characteristic function of $\sum_{n \leq x} \Lambda(n)/(n^{1/2+iU_T} \log n)$ (see [9, Theorem 2], where x has to be $O((\log T)^{2-\epsilon})$) or of the corresponding stochastic model (replace $\{p^{iU_T}\}_{p \in \mathbb{P}}$ by an independent sequence of random variables uniformly distributed on the unit circle, see [13, Example 4]).

In this paper we further investigate the distribution of the sum over primes $\sum_{p \leq x} p^{-1/2-it}$ as t varies in $[T, 2T]$ and its consequences on the distribution of values of the Riemann zeta-function on the critical line. Here, we will restrict ourselves to the case of $\operatorname{Im} \log \zeta(1/2 + it)$. Note that some of the arguments cannot be applied to the case of $\operatorname{Re} \log \zeta(1/2 + it)$. It is our first aim to establish mod-Gaussian convergence if x fulfills certain conditions. Precisely, in Sect. 4 we prove the following:

Theorem 1 *Let $x = e^{\log T/N}$ and N such that $x \rightarrow \infty$ and $N/\log \log T \rightarrow \infty$ as $T \rightarrow \infty$. Then*

$$e^{u^2(\log \log x + \gamma)/4} \frac{1}{T} \int_T^{2T} e^{iu \sum_{p \leq x} \frac{\sin(r \log p)}{\sqrt{p}}} dt \rightarrow \Phi(u) \quad \text{as } T \rightarrow \infty \quad (1.3)$$

locally uniformly for $u \in \mathbb{R}$. Here, γ denotes Euler's constant and Φ is the analytic function given by

$$\Phi(u) = \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p}\right)^{-u^2/4} J_0\left(\frac{u}{\sqrt{p}}\right), \quad (1.4)$$

where J_0 denotes the zeroth Bessel function (see, e.g., Sect. 3).

One interesting point of the result seems to be the size of x . It can be chosen large enough to obtain Selberg's central limit theorem with Selberg's explicit error term (see [19, Theorem 2] and "Appendix 1"). Moreover, we obtain the following improvement of (1.1):

Corollary 1 Assume RH. For T sufficiently large, we have

$$\frac{1}{T} \int_T^{2T} e^{iv \frac{\operatorname{Im} \log \zeta(1/2+it)}{\sqrt{(\log \log T)/2}}} dt = e^{-v^2/2} + v^2 O\left(\frac{\log \log \log T}{\log \log T}\right) + O(1/\log T)$$

uniformly for $|v| \leq \sqrt{\log \log T / \log \log \log T}$.

In Sect. 5 we deal with the question if the convergence in Theorem 1 can be extended to the complex plane. Assuming the Riemann hypothesis, we prove such a result for a weighted sum over primes.

Theorem 2 Assume RH. Let $x = e^{\log T/N}$ and N such that $x \rightarrow \infty$ and $N/\log \log T \rightarrow \infty$ as $T \rightarrow \infty$. Furthermore, let f be the function $f(u) = (\pi u/2) \cot(\pi u/2)$ and $\gamma_f = -0.1080 \dots$ be the constant defined by $\prod_{p \leq x} (1 - f^2(\log p / \log x) / p) = (e^{-\gamma_f} / \log x)(1 + o(1))$. Then

$$e^{z^2(\log \log x + \gamma_f)/4} \frac{1}{T} \int_T^{2T} e^{iz \sum_{p \leq x} \frac{\sin(r \log p)}{\sqrt{p}} f\left(\frac{\log p}{\log x}\right)} dt \rightarrow \Phi(z) \quad \text{as } T \rightarrow \infty$$

locally uniformly for $z \in \mathbb{C}$, where Φ is given by (1.4).

More general sums are possible as well (see [8, Lemma 1] and [2, Lemma 1]). For the evaluation of γ_f see [8, proof of Lemma 6].

The crucial step from Theorem 1 to Theorem 2 is an estimate of the exponential moments of the above sum. For this purpose let $x \leq T^2$ and $h \in \mathbb{R}$. Assuming the Riemann hypothesis, we then show that there exist constants C , C' , and C'' such that

$$\frac{1}{T} \int_T^{2T} e^{h \sum_{n \leq x} \frac{\Lambda(n)}{\log n} \frac{\sin(r \log n)}{\sqrt{n}} f\left(\frac{\log n}{\log x}\right)} dt \leq C'' e^{C|h| \frac{\log T}{\log x} + C'h^2 \log \log T}.$$

Note that this inequality, which is almost a subgaussian bound, is valid beyond the range which is contained in Theorems 1 and 2.

We turn to the applications of Theorem 2. As described above, Theorem 1 can be used to obtain results in connection with the central limit theorem. In addition, Theorem 2 yields large deviations results. Applying the Gärtner-Ellis theorem and Theorem 2, one obtains a large deviation principle (see [4, chapter 1.2] or “Appendix 3” for the definition of the large deviation principle) from which we will deduce the following two Corollaries.

Corollary 2 Assume RH. Let U_T be random variables uniformly distributed on $[T, 2T]$. Then the family $(1/((\log \log T)/2)) \operatorname{Im} \log \zeta(1/2 + iU_T)$ satisfies the large deviation principle with the speed $1/((\log \log T)/2)$ and the rate function $I(h) = h^2/2$. For instance,

$$\begin{aligned} & \frac{1}{(\log \log T)/2} \log \left(\frac{1}{T} \lambda(\{t \in [T, 2T] : \operatorname{Im} \log \zeta(1/2 + it) \geq h(\log \log T)/2\}) \right) \\ & \rightarrow -h^2/2 \quad \text{as } T \rightarrow \infty, \end{aligned} \tag{1.5}$$

where $h > 0$ and λ denotes the Lebesgue measure.

Corollary 3 Assume RH. Let $h \in \mathbb{R}$. Then

$$\frac{1}{(\log \log T)/2} \log \left(\frac{1}{T} \int_T^{2T} e^{h \operatorname{Im} \log \zeta(1/2+it)} dt \right) \rightarrow h^2/2 \quad \text{as } T \rightarrow \infty.$$

Related papers which also discuss large deviations results are the work of Radziwiłł [15], who extended the range of Selberg's central limit theorem for $\operatorname{Re} \log \zeta(1/2 + it)$ and the work of Soundararajan [20], who proved large deviation bounds for $\operatorname{Re} \log \zeta(1/2 + it)$. In fact, Soundararajan [20, Corollary A] completed the proof of Corollary 3 in the case of $\operatorname{Re} \log \zeta(1/2 + it)$ by proving the upper bound. The result can be stated as follows. For all $\epsilon > 0$ and all $h > 0$ we have $(\log T)^{h^2-\epsilon} \ll_{h,\epsilon} \int_T^{2T} |\zeta(1/2 + it)|^{2h} dt \ll_{h,\epsilon} (\log T)^{h^2+\epsilon}$. Note that the proof of the upper bound also applies to the case of $\operatorname{Im} \log \zeta(1/2 + it)$ and that we apply a slightly weaker upper bound in the proofs of Theorem 2 and Corollary 3.

Notation For $y \geq 2$ and a function $g : [0, 1] \rightarrow [0, 1]$, we define

$$\begin{aligned} \Sigma_{g,y}(t) &= \sum_{p \leq y} \frac{1}{p^{1/2+it}} g\left(\frac{\log p}{\log y}\right), \\ \Sigma_{g,y}^*(t) &= \sum_{n \leq y} \frac{\Lambda(n)}{\log n} \frac{1}{n^{1/2+it}} g\left(\frac{\log n}{\log y}\right), \end{aligned}$$

$$r_{g,y}(t) = \log \zeta(1/2 + it) - \Sigma_{g,y}(t), \quad \text{and} \quad r_{g,y}^*(t) = \log \zeta(1/2 + it) - \Sigma_{g,y}^*(t).$$

2 Moments of a sum over primes

Section 2 is devoted to some standard mean value calculations. In doing so, we will apply the following generalization of the mean value theorem of Montgomery and Vaughan contained in [16, Theorem 1.4.3] (see also [21, Lemma 3.1]). Let a_1, \dots, a_M and b_1, \dots, b_M be complex numbers, $M \geq 2$, and let $T > 0$. Then

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} \left(\sum_{m \leq M} a_m m^{-it} \right) \overline{\left(\sum_{m \leq M} b_m m^{-it} \right)} dt \\ &= \sum_{m \leq M} a_m \bar{b}_m + \theta \frac{2D}{T} \sqrt{\sum_{m \leq M} m |a_m|^2} \sqrt{\sum_{m \leq M} m |b_m|^2}, \end{aligned} \quad (2.1)$$

where θ depends on the various parameters but satisfies $|\theta| \leq 1$ and D is the universal constant in [16, Theorem 1.4.3].

Proposition 1 Let $x \geq 2$ and $T > 0$ be real numbers, k be a nonnegative integer, and p_1, \dots, p_n be the prime numbers not exceeding x . Then

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} \left(\sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \right)^{2k} dt \\ &= \frac{1}{2^{2k}} \binom{2k}{k} \sum_{\lambda_1 + \dots + \lambda_n = k} \left(\frac{k!}{\lambda_1! \dots \lambda_n!} \right)^2 p_1^{-\lambda_1} \dots p_n^{-\lambda_n} + \theta \frac{2D}{T} \sqrt{n^{2k} (2k)!} \end{aligned} \quad (2.2)$$

and $|(1/T) \int_T^{2T} (\sum_{p \leq x} \sin(t \log p) / \sqrt{p})^{2k+1} dt| \leq (2D/T) \sqrt{n^{2k+1} (2k+1)!}$ with $|\theta| \leq 1$ and D the constant in (2.1). Furthermore, the main term in (2.2) is bounded by $((2k)!/2^{2k} k!) (\sum_{p \leq x} 1/p)^k$.

Proof From $\sin(t \log p) = (p^{it} - p^{-it})/2i$, we obtain

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} \left(\sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \right)^k dt \\ &= \frac{1}{(2i)^k} \sum_{j=0}^k \binom{k}{j} \frac{(-1)^j}{T} \int_T^{2T} \left(\sum_{p \leq x} \frac{1}{p^{1/2+it}} \right)^j \left(\sum_{p \leq x} \frac{1}{p^{1/2+it}} \right)^{k-j} dt. \end{aligned} \quad (2.3)$$

For $j = 1, \dots, k$, the multinomial theorem yields

$$\left(\sum_{p \leq x} \frac{1}{p^{1/2+it}} \right)^j = \sum_{\lambda_1 + \dots + \lambda_n = j} \frac{j!}{\lambda_1! \dots \lambda_n!} (p_1^{-\lambda_1} \dots p_n^{-\lambda_n})^{1/2+it}. \quad (2.4)$$

If we plug in (2.4) into (2.3) with k replaced by $2k$, we obtain from (2.1) that

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} \left(\sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \right)^{2k} dt \\ &= \frac{1}{2^{2k}} \binom{2k}{k} \sum_{\lambda_1 + \dots + \lambda_n = k} \left(\frac{k!}{\lambda_1! \dots \lambda_n!} \right)^2 p_1^{-\lambda_1} \dots p_n^{-\lambda_n} \\ &+ \frac{\theta 2D}{2^{2k} T} \sum_{j=0}^{2k} \binom{2k}{j} \sqrt{\sum_{\lambda_1 + \dots + \lambda_n = j} \left(\frac{j!}{\lambda_1! \dots \lambda_n!} \right)^2} \sqrt{\sum_{\lambda_1 + \dots + \lambda_n = 2k-j} \left(\frac{(2k-j)!}{\lambda_1! \dots \lambda_n!} \right)^2} \end{aligned} \quad (2.5)$$

with $|\theta| \leq 1$. Applying $j! / (\lambda_1! \dots \lambda_n!) \leq j!$, $j = 0, \dots, 2k$, we bound the absolute value of the remainder by

$$\frac{2D}{2^{2k} T} \sum_{j=0}^{2k} \binom{2k}{j} \sqrt{n^j j! n^{2k-j} (2k-j)!} \leq \frac{2D}{T} \sqrt{n^{2k} (2k)!}. \quad (2.6)$$

The main term in (2.2) can be bounded similarly. As in (2.5) and (2.6), we also bound the $(2k+1)$ th moment. Note that there is no main term in this case. This completes the proof. \square

We want to compare these mean value estimates to some random variables expectations. Therefore, let X_1, X_2, \dots be an i.i.d. sequence of random variables uniformly distributed on the unit circle and let p_1, \dots, p_n be the primes not exceeding x . Then

$$\mathbb{E} \left[\left(\sum_{i=1}^n \frac{\text{Im } X_i}{\sqrt{p_i}} \right)^{2k} \right] = \frac{1}{2^{2k}} \binom{2k}{k} \sum_{\lambda_1 + \dots + \lambda_n = k} \left(\frac{k!}{\lambda_1! \dots \lambda_n!} \right)^2 p_1^{-\lambda_1} \dots p_n^{-\lambda_n} \quad (2.7)$$

and $\mathbb{E} \left[\left(\sum_{i=1}^n \text{Im } X_i / \sqrt{p_i} \right)^{2k+1} \right] = 0$. To prove this, we replace $\sin(t \log p)$ by $\text{Im } X_i$ and integration by expectation in (2.3) and (2.4) and then apply the formula $\mathbb{E}[X_1^{\lambda_1} \dots X_n^{\lambda_n} X_1^{-\mu_1} \dots X_n^{-\mu_n}] = 1$ if $\lambda_j = \mu_j$ for all $j = 1, \dots, n$ and $= 0$ else.

3 Bessel functions

The Bessel functions appear in the Fourier expansion of the function $e^{iz \sin \theta}$,

$$e^{iz \sin \theta} = \sum_{k=-\infty}^{\infty} J_k(z) e^{ik\theta}. \quad (3.1)$$

Explicitly the k th Bessel function $J_k(z)$ is given by

$$J_k(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{k+2n}}{n!(k+n)!} \quad (3.2)$$

for $k \geq 0$ and given by the relation $J_k(z) = (-1)^k J_{-k}(z)$ for $k < 0$ (for these and more facts about Bessel functions see, e.g., the book of Andrews, Askey and Roy [1]). This section is devoted to the following mod-Gaussian convergence result (compare to [10, Proposition 4.1]).

Proposition 2 *Let X_1, X_2, \dots be an i.i.d. sequence of random variables uniformly distributed on the unit circle and let p_1, p_2, \dots be the increasing sequence of all primes. Then*

$$e^{z^2(\log \log x + \gamma)/4} \mathbb{E} \left[e^{iz \sum_{j=1}^{\pi(x)} \frac{\operatorname{Im} X_j}{\sqrt{p_j}}} \right] \rightarrow \Phi(z) \quad \text{as } x \rightarrow \infty \quad (3.3)$$

locally uniformly for $z \in \mathbb{C}$. Here, γ denotes Euler's constant, $\pi(x)$ denotes the number of primes not exceeding x , and $\Phi(z)$ is given by (1.4).

Proof By (3.1), we have

$$\mathbb{E} [e^{iz \operatorname{Im} X_1}] = \frac{1}{2\pi} \int_0^{2\pi} e^{iz \sin \theta} d\theta = J_0(z). \quad (3.4)$$

Applying the independence of the X_j 's, (3.4), and finally Merten's formula $\prod_{p \leq x} (1 - 1/p) = (e^{-\gamma} / \log x)(1 + o(1))$, we obtain that the left hand side of (3.3) is equal to

$$e^{z^2(\log \log x + \gamma)/4} \prod_{p \leq x} J_0 \left(\frac{z}{\sqrt{p}} \right) = (1 + o(1))^{z^2/4} \prod_{p \leq x} \left(1 - \frac{1}{p} \right)^{-z^2/4} J_0 \left(\frac{z}{\sqrt{p}} \right).$$

It remains to show that the above product converges to $\Phi(z)$, locally uniformly for $z \in \mathbb{C}$. This follows from the fact that the product $\Phi(z)$ is normally convergent (see [6, Chapter IV.1, especially Remark IV.1.7]). This completes the proof. \square

Consider the random variables $\operatorname{Im} \Sigma_{1,x}(-U_T)$, U_T being random variables uniformly distributed on $[T, 2T]$. Note that we computed their moments in Proposition 1. As mentioned in the Introduction, one can use the method of moments to deduce that, as $x \rightarrow \infty$, $x = T^{o(1)}$, $(1/\sqrt{(\log \log x)/2}) \operatorname{Im} \Sigma_{1,x}(-U_T)$ converges in distribution to a Gaussian random variable with expectation 0 and variance 1 (see [2, Proof of Theorem B]). We will generalize this result by considering the cumulants of $\operatorname{Im} \Sigma_{1,x}(-U_T)$.

If Y is a real random variable such that $\mathbb{E}[e^{zY}]$ exists and is finite for all $z \in \mathbb{C}$, $\mathbb{E}[e^{zY}]$ is an analytic function and there exists a neighbourhood of 0 where $\log \mathbb{E}[e^{zY}] = \sum_{m=1}^{\infty} \kappa_m(Y) z^m / m!$. The coefficients $\kappa_m(Y)$, $m \geq 1$, are called the cumulants of Y . Thus, $\kappa_m(Y)$ is equal to the m th derivative of $\log \mathbb{E}[e^{zY}]$ evaluated at 0.

Corollary 4 Let $x = e^{\log T/N}$ and N such that $x \rightarrow \infty$ and $N \rightarrow \infty$ as $T \rightarrow \infty$ and let U_T be random variables uniformly distributed on $[T, 2T]$. Then, as $T \rightarrow \infty$, $\kappa_2(\text{Im } \Sigma_{1,x}(-U_T)) - (\log \log x + \gamma)/2 \rightarrow c_2$ and for $m \neq 2$ $\kappa_m(\text{Im } \Sigma_{1,x}(-U_T)) \rightarrow c_m$, where the c_m 's are defined by the series expansion $\log \Phi(-iz) = \sum_{m=1}^{\infty} c_m z^m/m!$, for z in a neighbourhood of 0.

Proof By the construction of Φ , there exists a real number $0 < r \leq 1$ such that for $|z| \leq r$, $\log \Phi(z) = \sum_{p \in \mathbb{P}} ((-z^2/4) \log(1 - 1/p) + \log J_0(z/\sqrt{p}))$. Hence, by Merten's formula,

$$(-z^2/4)(\log \log x + \gamma) + \sum_{p \leq x} \log J_0\left(\frac{-iz}{\sqrt{p}}\right) \rightarrow \log \Phi(-iz) \quad \text{as } x \rightarrow \infty \quad (3.5)$$

uniformly for $|z| \leq r, z \in \mathbb{C}$. The uniform convergence implies (see [6, Theorem III.1.3]), that the m th derivative of the left hand side of (3.5) evaluated at 0 converges to c_m . Hence, under the assumptions of Proposition 2, the cumulants of $\sum_{j=1}^{\pi(x)} \text{Im } X_j/\sqrt{p_j}$ satisfy the convergence described in Corollary 4, since $\mathbb{E}[\exp(z \sum_{j=1}^{\pi(x)} \text{Im } X_j/\sqrt{p_j})] = \prod_{p \leq x} J_0(-iz/\sqrt{p})$. It remains to show that for $m \geq 1$

$$\kappa_m(\text{Im } \Sigma_{1,x}(-U_T)) - \kappa_m\left(\sum_{j=1}^{\pi(x)} \frac{\text{Im } X_j}{\sqrt{p_j}}\right) \rightarrow 0 \quad \text{as } T \rightarrow \infty. \quad (3.6)$$

To prove this, we use the fact that the cumulants can be expressed in terms of the moments, namely $\kappa_m(Y) = \sum a_{\lambda_1, \dots, \lambda_m} \mathbb{E}[Y^1]^{\lambda_1} \dots \mathbb{E}[Y^m]^{\lambda_m}$, where the sum is over all positive integers such that $1\lambda_1 + 2\lambda_2 + \dots + m\lambda_m = m$, $a_{\lambda_1, \dots, \lambda_m}$ are integers, and Y is a random variable as above. If we plug in Proposition 1 and (2.7) into this formula, (3.6) follows from multiplying out since for $k \leq m$ and $x \geq 3$ the main terms in (2.2) are $O((\sum_{p \leq x} 1/p)^m) = O((\log \log x)^m)$ (see [3, (5) of chapter 7]), while for $k \leq m$ the remainders in (2.2) are $O(T^{(m/N)-1})$ which is $O(T^{-a})$ for some $0 < a < 1$ if T is sufficiently large. \square

4 Mod-convergence of a sum over primes

By means of Proposition 1 and (2.7), we can apply the method of moments for fixed x and obtain the following convergence

$$\frac{1}{T} \int_T^{2T} e^{iu \sum_{p \leq x} \frac{\sin(r \log p)}{\sqrt{p}}} dt \rightarrow \prod_{p \leq x} J_0\left(\frac{u}{\sqrt{p}}\right) \quad \text{as } T \rightarrow \infty. \quad (4.1)$$

Another proof of (4.1) is contained in [14, Theorem 5.1]. The techniques used therein can be applied to get Theorem 1 and Theorem 2 for the choice $x = (\log T)^{2-\epsilon}$, $\epsilon > 0$ arbitrary. The improvement of Theorem 1 follows from Proposition 3 combined with Proposition 2.

Proposition 3 Let $c > 1$ be a constant. Define $x = e^{\log T/N}$ with $N = (c'ec^2/4) \log \log T$, where $c' > 1$ is allowed to depend on T but such that $x \rightarrow \infty$ as $T \rightarrow \infty$. For $T \geq 3$, sufficiently large such that $x \geq 2$ and $N \geq 1$, we have

$$\frac{1}{T} \int_T^{2T} e^{iu \sum_{p \leq x} \frac{\sin(r \log p)}{\sqrt{p}}} dt = \prod_{p \leq x} J_0\left(\frac{u}{\sqrt{p}}\right) + O((1/c')^{N-1} + (2c^2/\log x)^N) \quad (4.2)$$

uniformly for $|u| \leq c, u \in \mathbb{R}$.

Proof of Theorem 1 We apply Proposition 3 with x, N as in Theorem 1 and c an arbitrary constant with $c > 1$. Since $c' \rightarrow \infty$ in that case, the remainder in (4.2) is $o(\exp(-c^2(\log \log T)/4))$. If we multiply in (4.2) both sides by $\exp(u^2(\log \log x + \gamma)/4)$ and then apply Proposition 2, we obtain (1.3) uniformly for $|u| \leq c$. Since $c > 1$ is arbitrary, this completes the proof. \square

Proof of Proposition 3 Let $N' = \lfloor N \rfloor$. From the Taylor expansion $e^{iu} = \sum_{k \leq 2N'-1} (iu)^k/k! + \theta u^{2N'}/(2N')!$, $u \in \mathbb{R}$, with $|\theta| \leq 1$, we obtain

$$\begin{aligned} \frac{1}{T} \int_T^{2T} e^{iu \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}}} dt &= \sum_{k \leq 2N'-1} \frac{(iu)^k}{k!} \frac{1}{T} \int_T^{2T} \left(\sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \right)^k dt \\ &\quad + \theta \frac{u^{2N'}}{(2N')!} \frac{1}{T} \int_T^{2T} \left(\sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \right)^{2N'} dt \end{aligned} \quad (4.3)$$

with $|\theta| \leq 1$. By Proposition 1, the remainder is

$$O \left(\frac{c^{2N'}}{N'!} \frac{1}{2^{2N'}} \left(\sum_{p \leq x} \frac{1}{p} \right)^{N'} + \frac{(c^2 \pi(x))^{N'}}{T} \right).$$

Using the bound $(N')! \geq (N'/e)^{N'}$, elementary results in the theory of primes, namely the formulas $\sum_{p \leq x} 1/p = \log \log x + c_1 + O(1/\log x)$ and $\pi(x) \leq 2x/\log x$, and finally $N' = \lfloor N \rfloor$, this is

$$O \left(\left(\frac{ec^2 \log \log T}{4N'} \right)^{N'} + \frac{(c^2 \pi(x))^N}{T} \right) = O \left(\left(\frac{1}{c'} \right)^{N-1} + \left(\frac{2c^2}{\log x} \right)^N \right).$$

Now, let X_1, X_2, \dots be an i.i.d. sequence of random variables uniformly distributed on the unit circle. By Proposition 1 and (2.7), the moments in (4.3) are equal to those of the stochastic model plus a remainder which is bounded by $(2D/T)\sqrt{(\pi(x))^k k!}$. The resulting remainders in (4.3), $k \leq 2N' - 1$, add up to $O((c^2 \pi(x))^N/T) = O((2c^2/\log x)^N)$. Hence, (4.3) is equal to

$$\sum_{k \leq 2N'-1} \frac{(iu)^k}{k!} \mathbb{E} \left[\left(\sum_{j=1}^{\pi(x)} \frac{\operatorname{Im} X_j}{\sqrt{p_j}} \right)^k \right] + O((1/c')^{N-1} + (2c^2/\log x)^N).$$

Applying the above Taylor expansion again, we obtain

$$\begin{aligned} \prod_{p \leq x} J_0 \left(\frac{u}{\sqrt{p}} \right) &= \mathbb{E} \left[e^{iu \sum_{j=1}^{\pi(x)} \frac{\operatorname{Im} X_j}{\sqrt{p_j}}} \right] \\ &= \sum_{k \leq 2N'-1} \frac{(iu)^k}{k!} \mathbb{E} \left[\left(\sum_{j=1}^{\pi(x)} \frac{\operatorname{Im} X_j}{\sqrt{p_j}} \right)^k \right] + \theta \frac{u^{2N'}}{(2N')!} \mathbb{E} \left[\left(\sum_{j=1}^{\pi(x)} \frac{\operatorname{Im} X_j}{\sqrt{p_j}} \right)^{2N'} \right] \end{aligned}$$

with $|\theta| \leq 1$. The remainder already appeared in (4.3) and is $O((1/c')^{N-1})$. This completes the proof. \square

5 Mod-convergence in the complex plane

Section 5 is devoted to the proof of Theorem 2. Here, we will apply an explicit formula obtained by Goldston [8, Lemma 1] assuming RH. For $4 \leq x \leq t^2$ and $t \neq \gamma$, we have

$$\begin{aligned} \operatorname{Im} \log \zeta(1/2 + it) &= - \sum_{n \leq x} \frac{\Lambda(n)}{\log n} \frac{\sin(t \log n)}{\sqrt{n}} f\left(\frac{\log n}{\log x}\right) \\ &\quad + \sum_{\gamma} \sin((t - \gamma) \log x) \int_0^{\infty} \frac{u}{u^2 + ((t - \gamma) \log x)^2} \frac{du}{\sinh u} \\ &\quad + O\left(\frac{1}{t(\log x)^2}\right), \end{aligned} \quad (5.1)$$

where $f(u) = (\pi u/2) \cot(\pi u/2)$. We will also apply the following estimate obtained by Soundararajan assuming RH. For every $h \in \mathbb{R}$ there exist constants $C', C'' > 0$ such that

$$\frac{1}{T} \int_T^{2T} e^{h \operatorname{Im} \log \zeta(1/2 + it)} dt \leq C'' e^{C' h^2 \log \log T}. \quad (5.2)$$

Soundararajan [20] proved (5.2) for $\operatorname{Re} \log \zeta(1/2 + it)$. However, by using [17, Theorem 1] instead of [20, Proposition], his arguments apply to $\operatorname{Im} \log \zeta(1/2 + it)$, too. We prove (compare to [2, Lemma 3 and Corollary]):

Proposition 4 *Assume RH. Let $4 \leq x \leq T^2$ and $f(u) = (\pi u/2) \cot(\pi u/2)$. For every $h \in \mathbb{R}$ there exist constants C, C' , and C'' such that*

$$\frac{1}{T} \int_T^{2T} e^{h \sum_{n \leq x} \frac{\Lambda(n)}{\log n} \frac{\sin(t \log n)}{\sqrt{n}} f\left(\frac{\log n}{\log x}\right)} dt \leq C'' e^{C|h| \frac{\log T}{\log x} + C' h^2 \log \log T}.$$

Proof of Theorem 2 The proof mainly differs from the proof of Theorem 1 and Proposition 3 in its estimation of the remainder term. Nevertheless, we will repeat the main steps. We assume that $|z| \leq c$, where $c > 1$ is an arbitrary constant and $z \in \mathbb{C}$, say $z = u - ih$ with $u, h \in \mathbb{R}$. Let $N' = \lfloor N/2 \rfloor$. From the Taylor expansion $e^{tz} = e^{h+iu} = \sum_{k \leq 2N'-1} (iz)^k/k! + \theta e^h (|z|^{2N'})/(2N')!$ with $|\theta| \leq 1$, we obtain

$$\begin{aligned} \frac{1}{T} \int_T^{2T} e^{iz \operatorname{Im} \Sigma_{f,x}(-t)} dt &= \sum_{k \leq 2N'-1} \frac{(iz)^k}{k!} \frac{1}{T} \int_T^{2T} (\operatorname{Im} \Sigma_{f,x}(-t))^k dt \\ &\quad + \theta \frac{c^{2N'}}{(2N')!} \frac{1}{T} \int_T^{2T} e^{h \operatorname{Im} \Sigma_{f,x}(-t)} (\operatorname{Im} \Sigma_{f,x}(-t))^{2N'} dt \end{aligned} \quad (5.3)$$

with $|\theta| \leq 1$. To continue as in the proof of Proposition 3, we use that under the assumptions of Proposition 1 we have

$$\frac{1}{T} \int_T^{2T} (\operatorname{Im} \Sigma_{f,x}(-t))^k dt = \mathbb{E} \left[\left(\sum_{j=1}^{\pi(x)} \frac{\operatorname{Im} X_j}{\sqrt{p_j}} f\left(\frac{\log p_j}{\log x}\right) \right)^k \right] + \theta \frac{2D}{T} \sqrt{(\pi(x))^k k!} \quad (5.4)$$

with $|\theta| \leq 1$. Moreover, if we replace k by $2k$, the main term is bounded by $((2k)!/2^{2k}k!)(\sum_{p \leq x} 1/p)^k$. These estimates follow as in the proof of Proposition 1 and (2.7). Applying the Cauchy–Schwarz inequality, the absolute value of the remainder in (5.3) can be bounded by

$$\frac{c^{2N'}}{(2N')!} \sqrt{\frac{1}{T} \int_T^{2T} (\operatorname{Im} \Sigma_{f,x}(-t))^{4N'} dt} \sqrt{\frac{1}{T} \int_T^{2T} e^{2h \operatorname{Im} \Sigma_{f,x}(-t)} dt}.$$

For $x \geq 2$, we have $|\operatorname{Im} \Sigma_{f,x}(-t) - \operatorname{Im} \Sigma_{f,x}^*(-t)| \leq (\log \log x)/2 + O(1)$, say $\leq CN$ for T sufficiently large. Hence, by (5.4) and Proposition 4, the above is

$$O \left(\frac{c^{2N'}}{(2N')!} \sqrt{\frac{(4N')!(\sum_{p \leq x} 1/p)^{2N'}}{2^{4N'}(2N')!}} + \frac{\sqrt{(4N')!(\pi(x))^{4N'}}}{T} \sqrt{e^{4CcN+4C'c^2N}} \right)$$

for T sufficiently large. Applying $(4N')!/(2^{4N'}(2N')!) \leq (2N')!$, $\sqrt{(2N')!} \geq (2N'/e)^{N'}$, $\pi(x) \leq 2x/\log x$, and $N' = \lfloor N/2 \rfloor$, there exists a constant $c'' > 0$ (depending on c , C , and C') such that this is

$$O \left(\left(\frac{c'' \sum_{p \leq x} 1/p}{N'} \right)^{N'} + \left(\frac{c''}{\log x} \right)^{N/2} \right).$$

Since $N'/\sum_{p \leq x} 1/p$ and $\log x$ tend to infinity, this is $o(\exp(-c^2(\log \log x)/4))$. Hence, applying (5.4) to the other terms, (5.3) is equal to

$$\sum_{k \leq 2N'-1} \frac{(iz)^k}{k!} \mathbb{E} \left[\left(\sum_{j=1}^{\pi(x)} \frac{\operatorname{Im} X_j}{\sqrt{p_j}} f \left(\frac{\log p_j}{\log x} \right) \right)^k \right] + o(e^{-c^2(\log \log T)/4})$$

uniformly for $|z| \leq c$. If we replace $\sin(t \log p)$ by $\operatorname{Im} X_i$ and integration by expectation in (5.3), we can bound the resulting remainder as above. In doing so, we apply (5.5) instead of Proposition 4. The result is that

$$\begin{aligned} \frac{1}{T} \int_T^{2T} e^{iz \operatorname{Im} \Sigma_{f,x}(-t)} dt &= \mathbb{E} \left[e^{iz \sum_{j=1}^{\pi(x)} \frac{\operatorname{Im} X_j}{\sqrt{p_j}} f \left(\frac{\log p_j}{\log x} \right)} \right] \\ &\quad + o(e^{-c^2(\log \log T)/4}) \end{aligned}$$

uniformly for $|z| \leq c$. If we multiply both sides by $\exp(z^2(\log \log x + \gamma_f)/4)$ and then apply the formula

$$e^{z^2(\log \log x + \gamma_f)/4} \mathbb{E} \left[e^{iz \sum_{j=1}^{\pi(x)} \frac{\operatorname{Im} X_j}{\sqrt{p_j}} f \left(\frac{\log p_j}{\log x} \right)} \right] \rightarrow \Phi(z) \quad \text{as } x \rightarrow \infty \quad (5.5)$$

locally uniformly for $z \in \mathbb{C}$, the statement of Theorem 2 follows by the same argument as in the proof of Theorem 1. (5.5) follows as in the proof of Proposition 2 by using the additional fact that

$$\prod_{p \leq x} \left(1 - \frac{1}{p} f^2 \left(\frac{\log p}{\log x} \right) \right)^{-z^2/4} J_0 \left(\frac{z}{\sqrt{p}} f \left(\frac{\log p}{\log x} \right) \right) \rightarrow \Phi(z) \quad \text{as } x \rightarrow \infty$$

locally uniformly for $z \in \mathbb{C}$. We conclude by a brief argument why this holds. Split the product in $p \leq y$ and $y < p \leq x$. The product over $y < p \leq x$ converges locally uniformly to 1 if $y \rightarrow \infty$, while one can show that the product over $p \leq y$, say $y = \log x$, converges locally uniformly to $\phi(z)$, by using, e.g., $f^2(\log p / \log x) - 1 = O((\log \log x) / \log x)$ if $p \leq \log x$. This completes the proof. \square

Proof of Proposition 4 From (5.1), (5.2), and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} e^{h \sum_{n \leq x} \frac{A(n)}{\log n} \frac{\sin(t \log n)}{\sqrt{n}}} f\left(\frac{\log n}{\log x}\right) dt \\ & \leq C''' e^{2C'h^2 \log \log T} \sqrt{\frac{1}{T} \int_T^{2T} e^{2h \sum_{\gamma} \sin((t-\gamma) \log x)} \int_0^\infty \frac{u}{u^2 + ((t-\gamma) \log x)^2} \frac{du}{\sinh u} dt} \end{aligned}$$

where C''' is a constant. The absolute value of the sum over zeros is bounded by a constant times

$$\sum_{|(t-\gamma) \log x| \leq 1} 1 + \sum_{|(t-\gamma) \log x| > 1} \frac{1}{((t-\gamma) \log x)^2} \quad (5.6)$$

and therefore it suffices to deal with the exponential moments of (5.6) with $h \geq 0$. Using the Cauchy–Schwarz inequality again, we obtain

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} e^{h \sum_{|(t-\gamma) \log x| \leq 1} 1 + h \sum_{|(t-\gamma) \log x| > 1} \frac{1}{((t-\gamma) \log x)^2}} dt \\ & \leq \sqrt{\frac{1}{T} \int_T^{2T} e^{2h \sum_{|(t-\gamma) \log x| \leq 1} 1} dt} \sqrt{\frac{1}{T} \int_T^{2T} e^{2h \sum_{|(t-\gamma) \log x| > 1} \frac{1}{((t-\gamma) \log x)^2}} dt}. \end{aligned}$$

We start with the first term, using the following fact on the number of zeros (see [3, (1) of Ch. 15])

$$N(t) = \frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O(1/t), \quad (5.7)$$

where $t \neq \gamma$ and $S(t) = (1/\pi) \operatorname{Im} \log \zeta(1/2 + it)$. We compute (note that $h \geq 0$)

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} e^{h \sum_{|(t-\gamma) \log x| \leq 1} 1} dt \\ & = \frac{1}{T} \int_T^{2T} e^{h \left(N\left(t + \frac{1}{\log x}\right) - N\left(t - \frac{1}{\log x}\right) \right)} dt \\ & \leq \frac{1}{T} \int_T^{2T} e^{Ch \frac{\log t}{\log x} + h \left(S\left(t + \frac{1}{\log x}\right) - S\left(t - \frac{1}{\log x}\right) \right)} dt \end{aligned}$$

$$\begin{aligned}
&\leq e^{Ch \frac{\log T}{\log x}} \sqrt{\frac{1}{T} \int_T^{2T} e^{2hS\left(t + \frac{1}{\log x}\right)} dt} \sqrt{\frac{1}{T} \int_T^{2T} e^{-2hS\left(t - \frac{1}{\log x}\right)} dt} \\
&= O\left(e^{Ch \frac{\log T}{\log x} + 4C'(h/\pi)^2 \log \log T}\right). \tag{5.8}
\end{aligned}$$

In the last step we used (5.2). Next, we divide the sum over $|(t - \gamma) \log x| > 1$ into $|t - \gamma| \geq T$, $1 < |t - \gamma| < T$, and $1/\log x < |t - \gamma| \leq 1$.

For $t \in [T, 2T]$, we have

$$\sum_{|t-\gamma| \geq T} \frac{1}{((t-\gamma) \log x)^2} = O\left(\sum_{\gamma} \frac{1}{\gamma^2 (\log x)^2}\right) = O\left(\frac{1}{(\log x)^2}\right).$$

The last step results from [3, (4) of Ch. 12]. For the second sum we use the fact that $N(t+1) - N(t) = O(1 + \log^+ |t|)$ (see [3, (2) of Ch. 15]). For $t \in [T, 2T]$, we obtain

$$\begin{aligned}
&\sum_{1 < |t-\gamma| < T} \frac{1}{((t-\gamma) \log x)^2} \\
&\leq \sum_{k=1}^{\lceil T \rceil - 1} \frac{N(t+k+1) - N(t+k)}{k^2 (\log x)^2} + \sum_{k=1}^{\lceil T \rceil - 1} \frac{N(t-k) - N(t-k-1)}{k^2 (\log x)^2} \\
&= O\left(\sum_{k=1}^{\lceil T \rceil - 1} \frac{\log T}{k^2 (\log x)^2}\right) = O\left(\frac{\log T}{(\log x)^2}\right).
\end{aligned}$$

Next, we consider the sum over $1/\log x < \gamma - t \leq 1$. We have

$$\sum_{1/\log x < \gamma - t \leq 1} \frac{1}{((t-\gamma) \log x)^2} \leq \sum_{j=1}^M \frac{N\left(t + \frac{k_j}{\log x}\right) - N\left(t + \frac{k_{j-1}}{\log x}\right)}{k_{j-1}^2} \tag{5.9}$$

where $1 = k_0 < k_1 < \dots < k_M$ with $k_{M-1} < \log x \leq k_M$. By (5.7), this is bounded by, recall $t \in [T, 2T]$,

$$\sum_{j=1}^M \left(\frac{C(k_j - k_{j-1}) \log T}{(\log x) k_{j-1}^2} + \frac{S\left(t + \frac{k_j}{\log x}\right) - S\left(t + \frac{k_{j-1}}{\log x}\right)}{k_{j-1}^2} \right).$$

We choose $k_j = 2^{j/2}$ and bound the left hand side of (5.9) by

$$\sqrt{2}C \frac{\log T}{\log x} + \sum_{j=1}^M \frac{S\left(t + \frac{2^{j/2}}{\log x}\right) - S\left(t + \frac{2^{(j-1)/2}}{\log x}\right)}{2^{j-1}}.$$

It follows that

$$\begin{aligned}
&\frac{1}{T} \int_T^{2T} e^{h \sum_{1/\log x < \gamma - t \leq 1} \frac{1}{((t-\gamma) \log x)^2}} dt \\
&\leq e^{\sqrt{2}Ch \frac{\log T}{\log x}} \frac{1}{T} \int_T^{2T} e^{h \sum_{j=1}^M \frac{1}{2^{j-1}} \left(S\left(t + \frac{2^{j/2}}{\log x}\right) - S\left(t + \frac{2^{(j-1)/2}}{\log x}\right) \right)} dt.
\end{aligned}$$

Using $\mathbb{E}[e^{h \sum_{j=1}^M X_j/2^j}] \leq \prod_{j=1}^M (\mathbb{E}[e^{h X_j}])^{1/2^j}$, which follows from repeated application of the Cauchy–Schwarz inequality, this is

$$\leq e^{\sqrt{2Ch} \frac{\log T}{\log x}} \prod_{j=1}^M \left(\frac{1}{T} \int_T^{2T} e^{2h \left(s \left(t + \frac{2^{j/2}}{\log x} \right) - s \left(t + \frac{2^{(j-1)/2}}{\log x} \right) \right)} dt \right)^{1/2^j}.$$

Applying again the Cauchy–Schwarz inequality and then (5.2) [as in (5.8)], this is

$$O \left(e^{\sqrt{2Ch} \frac{\log T}{\log x}} e^{16C'(h/\pi)^2 \log \log T} \right).$$

The same bound is true for the sum over $1/\log x < t - \gamma \leq 1$. The claim now follows from putting together all these estimates. \square

6 Proof of Corollary 1

Let T , c , c' , x , and N be as in Proposition 3, $T \geq 3$ sufficiently large such that $x \geq 2$ and $N \geq 2$. Assume further that $c' > 4$ is a constant such that the bound $(\log \log T)^{1/2} (c'/4)^{-N/2} = O(1/\log T)$ holds and that T is so big that the bound $(\log T)(2c^2/\log x)^{N/2} = O(1/\log T)$ holds, too. Then we show that

$$\begin{aligned} \frac{1}{T} \int_T^{2T} e^{iu \operatorname{Im} \log \zeta(1/2+it)} dt &= \prod_{p \leq x} J_0 \left(\frac{u}{\sqrt{p}} \right) \\ &\quad - \sum_{\substack{p \leq x \\ k \geq 3 \text{ odd}}} \frac{u}{k \sqrt{p}^k} J_k \left(\frac{u}{\sqrt{p}} \right) \prod_{\substack{q \leq x \\ q \neq p}} J_0 \left(\frac{u}{\sqrt{q}} \right) \\ &\quad + u^2 O(\log \log \log T) + O(1/\log T) \end{aligned} \quad (6.1)$$

uniformly for $|u| \leq c$, $u \in \mathbb{R}$. One can deduce Corollary 1 from (6.1) as follows. Replace u by $v/\sqrt{(\log \log T)/2}$ with $|v| \leq \sqrt{\log \log T / \log \log \log T}$ and let T be sufficiently large. Then, by (3.2) and the formulas $\sum_{p \leq x} 1/p = \log \log x + c_1 + O(1/\log x)$ and $\log \log x / \log \log T = 1 + O(\log \log \log T / \log \log T)$, the first term on the right hand side of (6.1) is equal to

$$\exp \left(\sum_{p \leq x} \log J_0(v/\sqrt{p(\log \log T)/2}) \right) = e^{-v^2/2} \left(1 + v^2 O \left(\frac{\log \log \log T}{\log \log T} \right) \right)$$

and, by using $|J_k(u)| \leq (|u|/2)^k/k!$ and $|J_0(u)| \leq 1$, $u \in \mathbb{R}$, the second term is $v^4 O(1/(\log \log T)^2)$ which is smaller than $v^2 O(\log \log \log T / \log \log T)$.

Hence, it remains to prove (6.1). From $\operatorname{Im} \log \zeta(1/2+it) = \operatorname{Im} \Sigma_{1,x}(t) + \operatorname{Im} r_{1,x}(t)$ and Taylor's theorem, we obtain

$$\begin{aligned} \frac{1}{T} \int_T^{2T} e^{iu \operatorname{Im} \log \zeta(1/2+it)} dt &= \frac{1}{T} \int_T^{2T} e^{iu \operatorname{Im} \Sigma_{1,x}(t)} dt + iu \frac{1}{T} \int_T^{2T} \operatorname{Im} r_{1,x}(t) e^{iu \operatorname{Im} \Sigma_{1,x}(t)} dt \\ &\quad + \theta \frac{u^2}{2} \frac{1}{T} \int_T^{2T} (\operatorname{Im} r_{1,x}(t))^2 dt \end{aligned}$$

with $|\theta| \leq 1$. By Proposition 3 and the above assumptions, the first term is equal to $\prod_{p \leq x} J_0(u/\sqrt{p}) + O(1/\log T)$ and by [21, Corollary of Theorem 5.1], the third term is $u^2 O(\log \log T)$. It remains to consider the second term. We start showing that

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} \operatorname{Im} \log \zeta(1/2 + it) e^{iu \operatorname{Im} \Sigma_{1,x}(t)} dt \\ &= \sum_{\substack{p \leq x \\ k \geq 1 \text{ odd}}} \frac{i}{k\sqrt{p}^k} J_k\left(\frac{u}{\sqrt{p}}\right) \prod_{\substack{q \leq x \\ q \neq p}} J_0\left(\frac{u}{\sqrt{q}}\right) + O(1/\log T) \end{aligned} \quad (6.2)$$

uniformly for $|u| \leq c$. Let $N' = \lfloor N/2 \rfloor$. From the Taylor expansion $e^{iu} = \sum_{k \leq 2N'-1} (iu)^k/k! + \theta u^{2N'}/(2N')!$, $u \in \mathbb{R}$, with $|\theta| \leq 1$, we obtain that the left hand side of (6.2) is equal to

$$\begin{aligned} & \sum_{k \leq 2N'-1} \frac{(iu)^k}{k!} \frac{1}{T} \int_T^{2T} \operatorname{Im} \log \zeta(1/2 + it) (\operatorname{Im} \Sigma_{1,x}(t))^k dt \\ &+ \theta \frac{c^{2N'}}{(2N')!} \frac{1}{T} \int_T^{2T} |\operatorname{Im} \log \zeta(1/2 + it)| (\operatorname{Im} \Sigma_{1,x}(t))^{2N'} dt \end{aligned} \quad (6.3)$$

with $|\theta| \leq 1$. Applying the Cauchy–Schwarz inequality, the estimates in the proof of Proposition 3, and $(1/T) \int_T^{2T} (\operatorname{Im} \log \zeta(1/2 + it))^2 dt = (\log \log T)/2 + O(1)$ (see [18, Theorem 3]), the remainder is $O((\log \log T)^{1/2}((c'/4)^{-N/2} + (2c^2/\log x)^{N/2}) = O(1/\log T)$. The remaining moments can be computed by using the following lemma which is a modification of [17, Lemma 5] and [8, equation (6.3)] and serves as a substitute for the mean value theorem of Montgomery and Vaughan in Sect. 2.

Lemma 1 *Assume RH. Let $k, h \leq T$ be two positive integers with $(k, h) = 1$. Then*

$$\begin{aligned} & \int_T^{2T} \log \zeta(1/2 + it) \left(\frac{k}{h}\right)^{it} dt = \frac{T \Lambda(k)}{\sqrt{k} \log k} + O(\sqrt{kh} \log T), \quad h = 1 \\ & \quad O(\sqrt{kh} \log T), \quad h \neq 1, \\ & \int_T^{2T} \operatorname{Im} \log \zeta(1/2 + it) \left(\frac{k}{h}\right)^{it} dt = \frac{-iT \Lambda(k)}{2\sqrt{k} \log k} + O(\sqrt{kh} \log T), \quad h = 1 \\ & \quad = \frac{iT \Lambda(h)}{2\sqrt{h} \log h} + O(\sqrt{kh} \log T), \quad k = 1 \\ & \quad O(\sqrt{kh} \log T), \quad h, k \neq 1. \end{aligned} \quad (6.4)$$

Denote by p_1, p_2, \dots, p_n the prime numbers not exceeding x . Let X_1, X_2, \dots be an i.i.d. sequence of random variables uniformly distributed on the unit circle. Furthermore, let $k, h \leq T$ be positive integers with $k/h = p_1^{-k_1} \dots p_n^{-k_n}$. Then (6.4) can be written as

$$\begin{aligned}
& \frac{1}{T} \int_T^{2T} \operatorname{Im} \log \zeta(1/2 + it) \left(p_1^{-k_1} \dots p_n^{-k_n} \right)^{it} dt \\
&= \mathbb{E} \left[- \sum_{j=1}^n \operatorname{Im} \log(1 - X_j / \sqrt{p_j}) X_1^{k_1} \dots X_n^{k_n} \right] + O \left(\frac{1}{T} \sqrt{p_1^{|k_1|} \dots p_n^{|k_n|}} \log T \right).
\end{aligned} \tag{6.5}$$

Expanding $(\operatorname{Im} \Sigma_{1,x}(t))^k$ as in (2.3) and (2.4), we deduce from (6.5) that

$$\begin{aligned}
& \frac{1}{T} \int_T^{2T} \operatorname{Im} \log \zeta(1/2 + it) (\operatorname{Im} \Sigma_{1,x}(t))^k dt \\
&= \mathbb{E} \left[\left(- \sum_{j=1}^n \operatorname{Im} \log(1 - X_j / \sqrt{p_j}) \right) \left(\sum_{j=1}^n \frac{\operatorname{Im} X_j}{\sqrt{p_j}} \right)^k \right] \\
&+ O \left(\frac{\log T}{2^k T} \sum_{l=0}^k \binom{k}{l} \sum_{\lambda_1 + \dots + \lambda_n = l} \frac{l!}{\lambda_1! \dots \lambda_n!} \sum_{\lambda_1 + \dots + \lambda_n = k-l} \frac{(k-l)!}{\lambda_1! \dots \lambda_n!} \right).
\end{aligned}$$

The remainder is $O((\log T)n^k/T)$ and the resulting remainders in (6.3), $k \leq 2N' - 1$, add up to $O((\log T)(2c/\log x)^N) = O(1/\log T)$. Hence, (6.3) is equal to

$$\begin{aligned}
& \sum_{k \leq 2N'-1} \frac{(iu)^k}{k!} \mathbb{E} \left[\left(- \sum_{j=1}^n \operatorname{Im} \log(1 - X_j / \sqrt{p_j}) \right) \left(\sum_{j=1}^n \frac{\operatorname{Im} X_j}{\sqrt{p_j}} \right)^k \right] + O(1/\log T) \\
&= \mathbb{E} \left[\left(- \sum_{j=1}^n \operatorname{Im} \log(1 - X_j / \sqrt{p_j}) \right) e^{iu \sum_{j=1}^n \frac{\operatorname{Im} X_j}{\sqrt{p_j}}} \right] \\
&+ \theta \frac{c^{2N'}}{(2N')!} \mathbb{E} \left[\left| - \sum_{j=1}^n \operatorname{Im} \log(1 - X_j / \sqrt{p_j}) \right| \left(\sum_{j=1}^n \frac{\operatorname{Im} X_j}{\sqrt{p_j}} \right)^{2N'} \right] + O(1/\log T).
\end{aligned} \tag{6.6}$$

The last equality follows from applying Taylor's theorem as in (6.3). If we treat the first remainder in the last row as the corresponding one in (6.3), using $\mathbb{E}[(\sum_{j=1}^n \operatorname{Im} \log(1 - X_j / \sqrt{p_j}))^2] = (\log \log x)/2 + O(1)$ this time, one can show that it is also $O(1/\log T)$. By plugging in (3.1) and expanding the logarithm, we obtain that (6.6) is equal to

$$\sum_{\substack{p \leq x \\ k \geq 1 \text{ odd}}} \frac{i}{k \sqrt{p}^k} J_k \left(\frac{u}{\sqrt{p}} \right) \prod_{\substack{q \leq x \\ q \neq p}} J_0 \left(\frac{u}{\sqrt{q}} \right) + O(1/\log T)$$

which completes the proof of (6.2). The last step in the proof of (6.1) is to show that

$$\frac{1}{T} \int_T^{2T} \operatorname{Im} \Sigma_{1,x}(t) e^{iu \operatorname{Im} \Sigma_{1,x}(t)} dt$$

$$\begin{aligned}
&= \mathbb{E} \left[\left(\sum_{j=1}^n \frac{\operatorname{Im} X_j}{\sqrt{p_j}} \right) e^{iu \sum_{j=1}^n \frac{\operatorname{Im} X_j}{\sqrt{p_j}}} \right] + O(1/\log T) \\
&= \sum_{p \leq x} \frac{i}{\sqrt{p}} J_1 \left(\frac{u}{\sqrt{p}} \right) \prod_{\substack{q \leq x \\ q \neq p}} J_0 \left(\frac{u}{\sqrt{q}} \right) + O(1/\log T)
\end{aligned}$$

uniformly for $|u| \leq c$. The first equality follows as above or as in the proof of Proposition 1, the second equality again by plugging in (3.1). This completes the proof.

7 Proof of Corollary 2 and 3

Proof of Corollary 2 Let $x \geq 2$ be as in Theorem 2 with the additional property that $N/\log \log T = O(\log \log T)$. By Theorem 2 and the fact that $\log \log T/\log \log x \rightarrow 1$ in this case, we obtain for each $h \in \mathbb{R}$

$$\frac{1}{(\log \log T)/2} \log \left(\frac{1}{T} \int_T^{2T} e^{h \operatorname{Im} \Sigma_{f,x}(t)} dt \right) \rightarrow h^2/2 \quad \text{as } T \rightarrow \infty. \quad (7.1)$$

Using Theorem 3, we obtain that the family $(1/((\log \log T)/2)) \operatorname{Im} \Sigma_{f,x}(U_T)$ satisfies the large deviation principle with the speed $1/((\log \log T)/2)$ and the rate function $I(h) = h^2/2$. Next, consider $\operatorname{Im} r_{f,x}(U_T)$. We will show that there exists a constant $C > 0$ (the constant in (7.4)) such that for each $\delta > 0$

$$\begin{aligned}
(1/T) \lambda(\{t \in [T, 2T] : |\operatorname{Im} r_{f,x}(t)| \geq C \delta \log \log T\}) \\
\leq e^{-(1-o(1))(\delta \log \log T) \log(\delta \log \log T)}. \quad (7.2)
\end{aligned}$$

We postpone the proof of (7.2) to the end of this section. From (7.2) we deduce that for each $\delta > 0$

$$\begin{aligned}
&\frac{1}{(\log \log T)/2} \log \left(\frac{1}{T} \lambda(\{t \in [T, 2T] : |\operatorname{Im} r_{f,x}(t)| \geq \delta \log \log T\}) \right) \\
&\leq -2(\delta/C)(1-o(1))(\log \log \log T + \log(\delta/C)).
\end{aligned}$$

As $T \rightarrow \infty$, the right hand side goes to $-\infty$. Hence, by Definition 1, the families $(1/((\log \log T)/2)) \operatorname{Im} \log \zeta(1/2 + iU_T)$ and $(1/((\log \log T)/2)) \Sigma_{f,x}(U_T)$ are exponentially equivalent. To obtain the statement of the theorem, we finally apply [4, Theorem 4.2.13], which states that if two families of random variables are exponentially equivalent, and one of them satisfies the large deviation principle with good rate function I , then the same large deviation principle holds for the other family.

It remains to show (7.2). Therefore, let $V = \delta \log \log T$ and decompose

$$\operatorname{Im} r_{f,x} = \operatorname{Im} \left(r_{g,T^{1/V}}^* + (\Sigma_{g,T^{1/V}}^* - \Sigma_{g,T^{1/V}}) + (\Sigma_{g,T^{1/V}} - \Sigma_{g,x}) + \Sigma_{g-f,x} \right).$$

If $|\operatorname{Im} r_{f,x}(t)| \geq CV$, there exists a summand on the right hand side whose absolute value is greater or equal to $CV/4$. Applying the union bound, we obtain

$$\begin{aligned}
&(1/T) \lambda(\{t \in [T, 2T] : |\operatorname{Im} r_{f,x}(t)| \geq CV\}) \\
&\leq (1/T) \lambda(\{t \in [T, 2T] : |\operatorname{Im} r_{g,T^{1/V}}^*(t)| \geq CV/4\}) \\
&\quad + (1/T) \lambda(\{t \in [T, 2T] : |\operatorname{Im} \Sigma_{g,T^{1/V}}^*(t) - \operatorname{Im} \Sigma_{g,T^{1/V}}(t)| \geq CV/4\})
\end{aligned}$$

$$\begin{aligned}
& + (1/T)\lambda(\{t \in [T, 2T] : |\operatorname{Im} \Sigma_{g,T^{1/V}}(t) - \operatorname{Im} \Sigma_{g,x}(t)| \geq CV/4\}) \\
& + (1/T)\lambda(\{t \in [T, 2T] : |\operatorname{Im} \Sigma_{g-f,x}(t)| \geq CV/4\}).
\end{aligned} \tag{7.3}$$

If we choose Selberg's function $g(u) = e^{-2u} \min(1, 2(1-u))$, we can apply [17, Theorem 1], which says that, assuming RH, there exists constants $C, C' > 0$ such that for $2 \leq y \leq t^2$ and $t \geq 2$,

$$|\operatorname{Im} r_{g,y}^*(t)| \leq \left| \frac{C'}{\log y} \sum_{n \leq y} \frac{\Lambda(n)}{n^{1/2+it}} g\left(\frac{\log n}{\log y}\right) \right| + \frac{C}{16} \frac{\log t}{\log y}. \tag{7.4}$$

If we choose $y = T^{1/V}$ and $t \in [T, 2T]$, $T \geq 2$, we have $(C/16)(\log t / \log y) \leq CV/8$. For $T \geq 2$, sufficiently large such that $2 \leq T^{1/V} \leq T^2$, we obtain

$$\begin{aligned}
& (1/T)\lambda(\{t \in [T, 2T] : |\operatorname{Im} r_{g,T^{1/V}}^*(t)| \geq CV/4\}) \\
& \leq \frac{1}{T}\lambda\left(\left\{t \in [T, 2T] : \left| \frac{C'}{\log T^{1/V}} \sum_{n \leq T^{1/V}} \frac{\Lambda(n)}{n^{1/2+it}} g\left(\frac{\log n}{\log T^{1/V}}\right) \right| \geq CV/8 \right\}\right).
\end{aligned}$$

Now, we can apply Markov's inequality and (9.1) to bound the last term by

$$\left(\frac{8C'}{CV}\right)^{2\lfloor V \rfloor} 3^{2V} (2(AV)^V + O(1)^V) = e^{-(1-o(1))V \log V}.$$

Similarly, by using the other bounds in "Appendix 2", we can bound the three other terms in (7.3) by $\exp(-(1-o(1))V \log V)$. Hence, (7.2) follows. This completes the proof. \square

Proof of Corollary 3 The asserted formula is exactly content of Varadhan's integral lemma (see Theorem 4). The assumptions of the theorem are satisfied by Corollary 2 and Eq. (5.2). \square

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8 Appendix 1: Selberg's result

In this appendix we briefly discuss Selberg's result about the rate of convergence in the central limit theorem of $\operatorname{Im} \log \zeta(1/2 + it)$ (see [19, Theorem 2] and [21, Theorem 6.2]). From Theorem 1 we deduce:

Lemma 2 Let $x = e^{\log T/N}$ and N such that $x \rightarrow \infty$ and $N/\log \log T \rightarrow \infty$ as $T \rightarrow \infty$. Suppose further that $N/\log \log T = O(\log \log T)$. Then

$$\begin{aligned}
& \sup_{a < b} \left(\frac{1}{T} \lambda \left(\left\{ t \in [T, 2T] : \frac{1}{\sqrt{(\log \log x + \gamma)/2}} \sum_{p \leq x} \frac{\sin(t \log p)}{\sqrt{p}} \in [a, b] \right\} \right) \right. \\
& \left. - \int_a^b e^{-t^2/2} \frac{dt}{\sqrt{2\pi}} \right) = O(1/\sqrt{\log \log T}).
\end{aligned} \tag{8.1}$$

Proof We denote by $\Phi_n(u)$ the left hand side of (1.3). Using [5, XVI.3, formula 3.13] we can bound the left hand side of (8.1) by

$$\frac{2}{\pi} \int_{-c\sqrt{\log \log x}}^{c\sqrt{\log \log x}} e^{-u^2/2} |(\Phi_n(u/\sqrt{(\log \log x + \gamma)/2}) - 1)/u| du + O\left(\frac{1}{c\sqrt{\log \log x}}\right). \quad (8.2)$$

An inspection of the proof of Proposition 2 combined with (4.2) shows that $\Phi_n(u) = \Phi(u)(1 + O(1/\log x)) + O(1/\log T)$, $|u| \leq c$. If we choose $c > 0$ such that $\Phi(u)$ has no zeros for $|u| \leq c$, we obtain $\Phi_n(u) = \Phi(u)(1 + O(1/\log x))$, $|u| \leq c$. On the other hand, we have $\Phi(u/\sqrt{(\log \log x + \gamma)/2}) = 1 + O(u^2/\log \log x)$, $|u| \leq c\sqrt{\log \log x}$. Plugging in these estimates gives that (8.2) is $O(1/\sqrt{\log \log x})$. From $N/\log \log T = O(\log \log T)$ we conclude that $\log \log T/\log \log x \rightarrow 1$ and this completes the proof. \square

This lemma combined with the bound (see [21, Lemma 6.2])

$$|\{t \in [T, 2T] : |r_{1,x}(t)| \geq c' \log \log \log T\}| = O(1/\sqrt{\log \log T}),$$

where $c' > 0$ is a constant, yields Selberg's result

$$\sup_{a < b} \left(\frac{1}{T} \lambda \left(\left\{ t \in [T, 2T] : \frac{\operatorname{Im} \log \zeta(1/2 + it)}{\sqrt{(\log \log T)/2}} \in [a, b] \right\} \right) - \int_a^b e^{-t^2/2} \frac{dt}{\sqrt{2\pi}} \right) = O\left(\frac{\log \log \log T}{\sqrt{\log \log T}}\right).$$

9 Appendix 2: Mean value estimates

For completeness we present some standard mean value estimates which we applied in the proof of Corollary 2 (see [17, Lemma 3] and [20, Lemma 3]). For this purpose let x and y be positive real numbers, a_p and b_p be complex numbers with $|a_p| \leq 1$ and $|b_p| \leq \log p/\log x$, and k be a nonnegative integer. By repeating the arguments in the proof of Proposition 1, we obtain

$$\begin{aligned} \frac{1}{T} \int_T^{2T} \left| \sum_{p \leq x} \frac{a_p}{p^{1+2it}} \right|^{2k} dt &\leq k! \left(\sum_{p \leq x} \frac{1}{p^2} \right)^k + 2Dk!(\pi(x))^k/T, \\ \frac{1}{T} \int_T^{2T} \left| \sum_{y < p \leq x} \frac{a_p}{p^{1/2+it}} \right|^{2k} dt &\leq k! \left(\sum_{y < p \leq x} \frac{1}{p} \right)^k + 2Dk!(\pi(x) - \pi(y))^k/T, \\ \frac{1}{T} \int_T^{2T} \left| \sum_{p \leq x} \frac{b_p}{p^{1/2+it}} \right|^{2k} dt &\leq k! \frac{1}{(\log x)^k} \left(\sum_{p \leq x} \frac{\log p}{p} \right)^k + 2Dk!(\pi(x))^k/T. \end{aligned}$$

If $x \leq T^{1/k}$, the first and the third term are bounded by $(Ak)^k$ and the second by $(k(\log \log x - \log \log y + A))^k$, $A > 0$ some constant.

For example, we obtain for a function $|g(u)| \leq 1$

$$\begin{aligned} & \frac{1}{T} \int_T^{2T} \left| \frac{1}{\log T^{1/V}} \sum_{n \leq T^{1/V}} \frac{\Lambda(n)}{n^{1/2+it}} g\left(\frac{\log n}{\log T^{1/V}}\right) \right|^{2\lfloor V \rfloor} dt \\ &= \frac{1}{T} \int_T^{2T} \left| \sum_{p \leq T^{1/V}} \frac{b_p}{p^{1/2+it}} + \sum_{p^2 \leq T^{1/V}} \frac{a_p}{p^{1+2it}} + O(1) \right|^{2\lfloor V \rfloor} dt \\ &\leq 3^{2V} ((AV)^V + (AV)^V + O(1)^V). \end{aligned} \quad (9.1)$$

10 Appendix 3: Large deviation theory

In this appendix we give the definition of the large deviation principle and state two important results which we used in the proofs of Corollary 2 and 3 (see [4]).

A function $I : \mathbb{R} \rightarrow [0, \infty]$ is called a rate function (resp. good rate function), if for all $\alpha \in [0, \infty)$, the sets $\{x : I(x) \leq \alpha\}$ are closed (resp. compact). A family $\{Z_\epsilon\}$ of real-valued random variables satisfies the large deviation principle with the speed ϵ and the rate function I , if

(a) For any closed set $F \subseteq \mathbb{R}$

$$\limsup_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P}(Z_\epsilon \in F) \leq - \inf_{x \in F} I(x).$$

(b) For any open set $G \subseteq \mathbb{R}$

$$\liminf_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P}(Z_\epsilon \in G) \geq - \inf_{x \in G} I(x).$$

Theorem 3 (Gärtner-Ellis, see Theorem 2.3.6 or 4.5.20 in [4]) *Suppose that for each $\lambda \in \mathbb{R}$*

$$\Lambda(\lambda) := \lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E}[e^{\lambda Z_\epsilon / \epsilon}]$$

exists and that Λ is differentiable. Then the family $\{Z_\epsilon\}$ satisfies the large deviation principle with the good rate function $I(x) = \sup_{\lambda \in \mathbb{R}} (\lambda x - \Lambda(\lambda))$.

Theorem 4 (Varadhan, see Theorem 4.3.1 in [4]) *Suppose that $\{Z_\epsilon\}$ satisfies the large deviation principle with a good rate function I and let $h \in \mathbb{R}$. Assume further that for some $\gamma > 1$*

$$\limsup_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E}[e^{\gamma h Z_\epsilon / \epsilon}] < \infty. \quad (10.1)$$

Then

$$\lim_{\epsilon \rightarrow 0} \epsilon \log \mathbb{E}[e^{h Z_\epsilon / \epsilon}] = \sup_{x \in \mathbb{R}} (xh - I(x)).$$

Definition 1 (see Definition 4.2.10 in [4]) Let $\{Z_\epsilon\}$ and $\{\tilde{Z}_\epsilon\}$ be two families of real-valued random variables, defined on the same probability space. Then $\{Z_\epsilon\}$ and $\{\tilde{Z}_\epsilon\}$ are called exponentially equivalent if for each $\delta > 0$,

$$\limsup_{\epsilon \rightarrow 0} \epsilon \log \mathbb{P}(|Z_\epsilon - \tilde{Z}_\epsilon| > \delta) = -\infty. \quad (10.2)$$

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